Lecture 4: Brauer Character Tables and Decomposition Matrices

Goal: Learn how to construct Brauer character tables using p-regular conjugacy classes, understand how ordinary characters restrict modulo p, and see how to compute and interpret decomposition matrices.

1. p-Regular Conjugacy Classes

Definition 4.1. Let G be a finite group and p a prime. An element $g \in G$ is called:

- p**-regular** if its order is not divisible by p,
- p**-singular** if its order is divisible by p.

Definition 4.2. A p**-regular class** is a conjugacy class consisting of p-regular elements.
Lemma 4.3. Brauer characters are defined only on p-regular classes.
Corollary 4.4. The number of irreducible modular representations (Brauer characters) of G over $\overline{\mathbb{F}}_p$ is equal to the number of p-regular conjugacy classes in G.

2. Brauer Character Tables

Definition 4.5. A **Brauer character table** is a table where:

- Rows: irreducible Brauer characters (modular irreducibles),
- Columns: p-regular conjugacy classes of G,
- Entries: character values in $\overline{\mathbb{F}}_p$ or in a minimal field of definition.

Notation: We often denote Brauer characters by $\varphi_1, \varphi_2, \ldots$, in contrast to $\chi_1, \chi_2, \ldots$ for ordinary irreducibles.

3. Decomposition Matrices

Definition 4.6 (Decomposition Matrix). Let $\{\chi_i\}$ be the ordinary irreducible characters of G, and $\{\varphi_j\}$ the Brauer characters. The **decomposition matrix** $D = (d_{ij})$ is defined by:

$$\chi_i|_{p\text{-reg}} = \sum_j d_{ij}\varphi_j,$$

where $d_{ij} \in \mathbb{Z}_{\geq 0}$ is the multiplicity of φ_j in the reduction of the representation affording χ_i.
Proposition 4.7. The decomposition matrix D has:

- Non-negative integer entries,
- One row per ordinary irreducible character,
- One column per irreducible Brauer character,
- Rank equal to the number of irreducible Brauer characters.

4. How to Compute Decomposition Matrices

- Identify all p-regular conjugacy classes in G.
- Construct a table of ordinary irreducible characters restricted to those classes.
- Write each restriction as a linear combination of unknown Brauer characters.
- Solve the resulting integer system (often using orthogonality relations or GAP).

Key Strategy: Use class function orthogonality and known values of modular characters to reconstruct the decomposition matrix.

5. Examples

Example 4.8: S_3, $p = 3$

Ordinary character table:

	[1]	[(12)]	[(123)]
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Only two 3-regular classes: identity and transpositions.

Brauer character table:

	[1]	[(12)]
φ_1	1	1
φ_2	2	0

Decomposition matrix:

	φ_1	φ_2
χ_1	1	0
χ_2	0	1
χ_3	1	1

6. Counterexamples

Counterexample 4.9. The decomposition matrix is not always invertible or square. For example, if a group has more ordinary characters than modular irreducibles, the matrix is rectangular.

7. Summary

In this lecture we learned:

- The concept of p-regular elements and their role in modular representation theory
- How Brauer character tables are built
- How to compute decomposition matrices by comparing ordinary and modular character data

Coming Up in Lecture 5: We will study *extensions of scalars*, the *Wedderburn decomposition* of group algebras in positive characteristic, and how these structures inform representation multiplicity and block theory.